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Short Communication

Dirac spinor's transformation under Lorentz mappings

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Abstract

For a given Lorentz matrix, we deduce the Dirac spinor's transformation in terms of four complex quantities.

Introduction

We have the Dirac equation for spin-1/2 particles [1-5] $\left[\left(x^{\mu}\right)=\left(t,x,y,z\right),\hbar=c=1\right]$:

$$\left(i\gamma^{\mu}\partial_{\mu}-m_{0}\right)\psi=0,\qquad i=\sqrt{-1},\qquad \partial_{\mu}=\frac{\partial}{\partial x^{\mu}}, \tag{1}$$

where ψ is a 4-spinor with the γ^{μ} matrices verifying the anticommutator [6-8]:

$$\left\{\gamma^{\mu},\gamma^{\nu}\right\}=2g^{\mu\nu}I_{4x4},\qquad \left(g^{\mu\nu}\right)=Diag\left(1,-1,-1,-1\right). \tag{2}$$

Here we shall use the Dirac-Pauli (or standard) representation [2,9]:

$$\gamma^{0}=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\qquad \gamma^{j}=\begin{pmatrix} 0 & \sigma_{j}\\ -\sigma_{j} & 0\end{pmatrix},\ j=1,2,3, \tag{3}$$

with the Cayley [10]-Sylvester [11]-Pauli [12] matrices:

$$\sigma_{1}=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\qquad \sigma_{2}=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix},\qquad \sigma_{3}=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}, \tag{4}$$

to analyze the transformation law of ψ under the orthochronic and proper Lorentz group [13-19]:

$$\tilde{x}^{\mu}=L^{\mu}{}_{\nu}\,x^{\nu}, \tag{5}$$

which implies the existence [2,7,20,21] of a non-singular matrix S such that:

$$L^{\mu}{}_{\nu}\,S\,\gamma^{\nu}=\gamma^{\mu}S, \tag{6}$$

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and we deduce the relativistic invariance of (1) if the Dirac 4-spinor obeys the transformation rule:

$$\tilde{\psi} = S\psi \ . \tag{7}$$

Here we exhibit a method to find$S$ for a given Lorentz matrix.

Construction of the matrix S for a given Lorentz mapping.

The arbitrary complex quantities α, β, γ, δ verifying the constraint $\alpha\delta - \beta\gamma = 1$, generate a Lorentz matrix $L = (L^{\mu}{}_{\nu})$ via the relations [13-15, 17, 22-26]:

$$\begin{aligned}
&L^0{}_0 = \frac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}\right), \quad L^1{}_0 = \frac{1}{2}\left(\alpha\bar{\gamma} + \beta\bar{\delta}\right) + cc, \quad L^2{}_0 = -\frac{i}{2}\left(\alpha\bar{\gamma} - \beta\bar{\delta}\right) + cc,\\
&L^0{}_1 = \frac{1}{2}\left(\alpha\bar{\beta} + \gamma\bar{\delta}\right) + cc, \quad L^1{}_1 = \frac{1}{2}\left(\alpha\bar{\delta} + \beta\bar{\gamma}\right) + cc, \quad L^2{}_1 = -\frac{i}{2}\left(\alpha\bar{\delta} + \beta\bar{\gamma}\right) + cc,\\
&L^0{}_2 = -\frac{i}{2}\left(\alpha\bar{\beta} + \gamma\bar{\delta}\right) + cc, \quad L^1{}_2 = -\frac{i}{2}\left(\alpha\bar{\delta} + \beta\bar{\gamma}\right) + cc, \quad L^2{}_2 = \frac{1}{2}\left(\alpha\bar{\delta} - \beta\bar{\gamma}\right) + cc,\\
&L^0{}_3 = \frac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}\right), \quad L^1{}_3 = \frac{1}{2}\left(\alpha\bar{\gamma} - \beta\bar{\delta}\right) + cc, \quad L^2{}_3 = -\frac{i}{2}\left(\alpha\bar{\gamma} + \beta\bar{\delta}\right) + cc,\\
&L^3{}_0 = \frac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}\right), \quad L^3{}_1 = \frac{1}{2}\left(\alpha\bar{\beta} - \gamma\bar{\delta}\right) + cc, \quad L^3{}_2 = -\frac{i}{2}\left(\alpha\bar{\beta} - \gamma\bar{\delta}\right) + cc,\\
&L^3{}_3 = \frac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}\right), \quad \alpha\delta - \beta\gamma = 1,
\end{aligned} \tag{8}$$

where cc means the complex conjugate of all the previous terms.

The inverse problem is to obtain α, β, γ, δ if we know L, and the answer is [26-29]:

$$\begin{aligned}
\alpha &= \frac{1}{D}Q^1{}_1 = \frac{1}{2D}\left[L^0{}_0 + L^0{}_3 + L^1{}_1 + L^2{}_2 + L^3{}_0 + L^3{}_3 - i\left(L^1{}_2 - L^2{}_1\right)\right],\\
\beta &= \frac{1}{D}Q^1{}_2 = \frac{1}{2D}\left[L^0{}_1 + L^1{}_0 - L^1{}_3 + L^3{}_1 + i\left(L^0{}_2 + L^2{}_0 - L^2{}_3 + L^3{}_2\right)\right],\\
\gamma &= \frac{1}{D}Q^2{}_1 = \frac{1}{2D}\left[L^0{}_1 + L^1{}_0 + L^1{}_3 - L^3{}_1 - i\left(L^0{}_2 + L^2{}_0 + L^2{}_3 - L^3{}_2\right)\right],\\
\delta &= \frac{1}{D}Q^2{}_2 = \frac{1}{2D}\left[L^0{}_0 - L^0{}_3 + L^1{}_1 + L^2{}_2 - L^3{}_0 + L^3{}_3 + i\left(L^1{}_2 - L^2{}_1\right)\right],
\end{aligned} \tag{9}$$

where $D^2 = Q^1{}_1 Q^2{}_2 - Q^1{}_2 Q^2{}_1$

From (6) are immediate the expressions [3, 30]:

$$L^{\mu}{}_0 = \frac{1}{4}tr\left(\gamma^0 S^{-1}\gamma^{\mu} S\right), \qquad L^{\mu}{}_k = -\frac{1}{4}tr\left(\gamma^k S^{-1}\gamma^{\mu} S\right), \quad \mu = 0,\ldots,3, \quad k = 1,2,3, \tag{10}$$

that is, if we know S then with (10) we can determine the Lorentz matrix; (10) generates the relations:

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$$L^0{}_0 = 2\left(b_0^2 - b_1^2 - b_2^2 - b_3^2\right) - 1, \qquad L^0{}_1 = 2\left[\left(b_2\, d_3 - b_3\, d_2\right) + i\left(b_0\, d_1 - b_1\, d_0\right)\right],$$
$$L^0{}_2 = 2\left[\left(b_3\, d_1 - b_1\, d_3\right) + i\left(b_0\, d_2 - b_2\, d_0\right)\right], \quad L^0{}_3 = 2\left[\left(b_1\, d_2 - b_2\, d_1\right) + i\left(b_0\, d_3 - b_3\, d_0\right)\right],$$
$$L^1{}_0 = 2\left[-\left(b_2\, d_3 - b_3\, d_2\right) + i\left(b_0\, d_1 - b_1\, d_0\right)\right], \quad L^1{}_1 = 2\left[\left(b_0^2 - b_1^2\right) + \left(d_2^2 + d_3^2\right)\right] - 1,$$
$$L^1{}_2 = 2\left[-\left(b_1\, b_2 + d_1\, d_2\right) - i\left(b_0\, b_3 + d_0\, d_3\right)\right], \quad L^1{}_3 = 2\left[-\left(b_1\, b_3 + d_1\, d_3\right) + i\left(b_0\, b_2 + d_0\, d_2\right)\right],$$
$$L^2{}_0 = 2\left[-\left(b_3\, d_1 - b_1\, d_3\right) + i\left(b_0\, d_2 - b_2\, d_0\right)\right], \quad L^2{}_1 = 2\left[-\left(b_1\, b_2 + d_1\, d_2\right) + i\left(b_0\, b_3 + d_0\, d_3\right)\right],$$
$$L^2{}_2 = 2\left[\left(b_0^2 - b_2^2\right) + \left(d_1^2 + d_3^2\right)\right] - 1, \qquad L^2{}_3 = 2\left[-\left(b_2\, b_3 + d_2\, d_3\right) - i\left(b_0\, b_1 + d_0\, d_1\right)\right],$$
$$L^3{}_0 = 2\left[-\left(b_1\, d_2 - b_2\, d_1\right) + i\left(b_0\, d_3 - b_3\, d_0\right)\right], \quad L^3{}_1 = 2\left[-\left(b_1\, b_3 + d_1\, d_3\right) - i\left(b_0\, b_2 + d_0\, d_2\right)\right],$$
$$L^3{}_2 = 2\left[-\left(b_2\, b_3 + d_2\, d_3\right) + i\left(b_0\, b_1 + d_0\, d_1\right)\right], \quad L^3{}_3 = 2\left[\left(b_0^2 - b_3^2\right) + \left(d_1^2 + d_2^2\right)\right] - 1, \tag{11}$$

which allow to obtain L if we have the expansion [31]:

$$S = b_0\, I + i d_0\, \gamma^5 + b_1\, \sigma^{23} + b_2\, \sigma^{31} + b_3\, \sigma^{12} + \sum_{j=1}^{3} d_j \sigma^{oj}. \tag{12}$$

However, here we have the inverse problem, that is, to obtain $b_\mu \,\&\, d_\mu, \mu = 0,\ldots,3$ verifying (11) for a given Lorentz matrix. Our answer is the following:

$$b_0 = \frac{1}{4}\left(\alpha + \bar{\alpha} + \delta + \bar{\delta}\right),\ b_1 = \frac{1}{4}\left(\bar{\beta} - \beta + \bar{\gamma} - \gamma\right),\ b_2 = \frac{i}{4}\left(\beta + \bar{\beta} - \gamma - \bar{\gamma}\right),\ b_3 = \frac{1}{4}\left(\bar{\alpha} - \alpha + \delta - \bar{\delta}\right),$$
$$d_0 = \frac{i}{4}\left(\alpha - \bar{\alpha} + \delta - \bar{\delta}\right),\ d_1 = -\frac{i}{4}\left(\bar{\beta} + \beta + \bar{\gamma} + \gamma\right),\ d_2 = \frac{1}{4}\left(\bar{\beta} - \beta + \gamma - \bar{\gamma}\right),\ d_3 = \frac{i}{4}\left(\bar{\delta} + \delta - \alpha - \bar{\alpha}\right), \tag{13}$$

hence the expressions (8) are deduced if we apply (13) into (11). Besides, with (13) the matrix (12) acquires the structure:

$$S = \begin{pmatrix} A & E \\ E & A \end{pmatrix}, \qquad A = \frac{1}{2}\begin{pmatrix} \bar{\alpha} + \delta & \bar{\beta} - \gamma \\ \bar{\gamma} - \beta & \alpha + \bar{\delta} \end{pmatrix}, \qquad E = \frac{1}{2}\begin{pmatrix} \bar{\alpha} - \delta & \bar{\beta} + \gamma \\ \bar{\gamma} + \beta & \bar{\delta} - \alpha \end{pmatrix}. \tag{14}$$

Therefore, for a given Lorentz transformation first we employ (9) to determine α, β, γ, δ, then S is immediate via (14); this approach is an alternative to the process showed in [31] and to the explicit general formula obtained by Macfarlane [30]:

$$\mathrm{S} = \frac{1}{4\sqrt{\mathrm{G}}}\ [\mathrm{G\,I} + \frac{\mathrm{i}}{2}\ \varepsilon_{\mu\nu\alpha\beta}\ \mathrm{L}^{\mu\nu}\ \mathrm{L}^{\alpha\beta}\ \gamma^5 + \mathrm{i}\,\Gamma\left(\mathrm{L}^2\right) - \mathrm{i}\left(2 + \mathrm{tr\,L}\right)\Gamma\left(\mathrm{L}\right)] \tag{15}$$

$$G = 2(1 + tr\,L) + \frac{1}{2}\left[\ (tr\,L)^2 - tr\,L^2\ \right], \qquad tr\,L = \sum_{\mu=0}^{3} L^\mu{}_\mu, \qquad tr\,L^2 = \sum_{\nu,\alpha=0}^{3} L^\nu{}_\alpha L^\alpha{}_\nu\ ,$$
$$\Gamma(L) = \sum_{\mu,\nu=0}^{3} L_{\mu\nu}\sigma^{\mu\nu}, \qquad \Gamma\left(L^2\right) = \sum_{\alpha,\mu,\nu=0}^{3} L_{\mu\alpha}L^\alpha{}_\nu\sigma^{\mu\nu}, \tag{16}$$

however, the possible physical applications are not evident in it. In our procedure, for example, the relations (9) are of great interest for the physicists working on supersymmetry [29], and the expressions (14) are very useful to study the relativistic motion of a classical point particle [28].

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Conclusion

The Dirac equation is relativistic if the corresponding 4-spinor verifies the transformation (7) under Lorentz mappings, with the matrix S satisfying the condition (6). Here we showed a procedure to construct S for a given Lorentz matrix.

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