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Research Article

Vibration eigenfrequencies of an elastic sphere with a large radius

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Abstract

An estimation is given for the free vibration eigenfrequencies (normal modes) of a homogeneous solid sphere with a large radius, with application to Earth's free vibrations. The free vibration eigenfrequencies of a fluid sphere are also derived as a particular case. Various corrections arising from static and dynamic gravitation, rotation, and inhomegeneities are estimated, and a tentative notion of an earthquake temperature is introduced.

Introduction

The long-standing interest in the vibrations of a solid sphere is related to the seismic vibrations of the Earth [1,2]. After a relatively short burst of energy in an earthquake the Earth continues to vibrate freely for a long time. Though with a liquid outer core and a viscous mantle, the Earth is still approximated by a solid sphere. Great progress in studying the vibrations of a homogeneous and isotropic elastic sphere has been made since the early days when Lamb introduced the vector spherical harmonics (Hansen vectors) [3-5]. The relevant eigenfrequencies were computed numerically as early as 1898 [6]. We discuss in this paper a natural simplification of this problem, which arises from the fact that a large radius of the sphere is a natural cutoff. Apart from giving formally the general solution of vibrations generated by the seismic tensorial force, we show that a large radius simplifies appreciably the boundary conditions, leading readily to the estimation of the eigenfrequencies (normal modes). The particular case of a fluid sphere is treated to a larger extent.

As it is well known, the vibrations of the Earth following an earthquake are of great importance in revealing the inner structure of the crust, mantle, and even the inner cores of the Earth. These vibrations imply a large number of modes, usually classified as spheroidal and toroidal, with periods in a wide range from 10^{-3} - 10^{-4} to hours. They attenuate slowly in time, leading to the thermalization of the residual energy of the earthquake. Usually, they are studied numerically, from the recorded data. We give in this paper a thorough description of the vibrations of a sphere in the limit of a large radius, which simplifies greatly the problem. This simplification allows us to perform analytical calculations to a great extent. First, we present the eigenvibrations for a homogeneous and isotropic elastic solid sphere, with general boundary conditions. Second, we introduce the assumption of a large radius, as appropriate for Earth's vibrations. We show that the toroidal vibrations are easily amenable to analytical calculations, while useful quantitative estimations can be made for the spheroidal vibrations. As a useful example, we include the analysis of the vibrations of a fluid sphere. Further on, we investigate the effects of static and dynamic gravitation, a problem with a higher degree of difficulty. Also, the effects of the Coriolis and centrifugal forces are analyzed, with emphasis on their well-known frequency splitting. Finally, we discuss the

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possibility of estimating the temperature gained by the Earth, as a consequence of the thermalization energy released in an earthquake.

Solid sphere

The elastic vibrations of a homogeneous and isotropic solid are described by the equation

$$\mu curl\, curl\mathbf{u} - (\lambda + 2\mu) grad\, div\mathbf{u} - \rho\omega^2 \mathrm{u} = \mathbf{F}(\omega)\,, \tag{1}$$

where $\boldsymbol{u}$ is the local displacement, ρ is the density, μ and λ are the Lame elastic moduli, ω is the frequency, and $F(\omega)$ is the force [7]. The components of the seismic tensorial force are

$$F_i(\omega) = M_{ij}(\omega)\partial_j \delta(\mathrm{r} - \mathrm{r}_0)\,, \tag{2}$$

where $M_{ij}(\omega)$ is the Fourier transform of the seismic moment, $\mathbf{r}_0$ is the position of the point where the force is placed and i, j, ... = 1,2,3 are cartesian labels [8,9]. An equivalent form of equation (1) is

$$c_2^2 \Delta \mathrm{u} + (c_1^2 - c_2^2) grad\, div\mathrm{u} + \omega^2 \mathrm{u} = -\mathrm{f}\,, \tag{3}$$

where $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ is the velocity of the longitudinal elastic waves, $c_2 = \sqrt{\mu/\rho}$ is the velocity of the transverse elastic waves, and $f(\omega) = f(\omega) / \rho$ (also, $m_{ij}(\omega) = m_{ij}\ (\omega) / \rho$). As it is well known, equation (3) is separated into two inhomogeneous Helmholtz equations

$$c_1^2 \Delta\Phi + \omega^2\Phi = -\varphi\,, c_2^2 \Delta \mathrm{A} + \omega^2 \mathrm{A} = -\mathrm{h}\,, \tag{4}$$

by $\mathbf{u} = grad\Phi + curl\mathbf{A}$, $divh = 0$, $\mathbf{f} = grad\varphi + curl\mathbf{h}$, $divh = 0$,

where φ and h are given by $\Delta\varphi = div\mathrm{f}$, $\Delta\mathrm{h} = -curl\mathrm{f}$ (Helmholtz potentials). We get

$$\varphi = -\frac{1}{4\pi} m_{ij}\partial_i\partial_j \frac{1}{|\mathrm{r} - \mathrm{r}_0|}, h_i = \frac{1}{4\pi}\varepsilon_{ijk} m_{kl}\partial_j\partial_l \frac{1}{|\mathrm{r} - \mathrm{r}_0|} \tag{5}$$

(where ε_{ijk} is the antisymmetric tensor of rank three), such that we are led to consider the equation

$$c^2 \Delta F + \omega^2 F = \frac{1}{r} \tag{6}$$

with solution

$$F(r) = \frac{1 - \cos kr}{\omega^2 r}, k^2 = \omega^2 / c^2\,; \tag{7}$$

This solution results immediately from the vibration Green function $G = -\frac{\cos kr}{4\pi c^2 r}$ of the Helmholtz equation $c^2 \Delta G + \omega^2 G = \delta(\mathrm{r})$. We get a particular solution of equation (3)

$$u_i^p = \frac{1}{4\pi} m_{ij}\partial_j \Delta F_2(|\mathbf{r} - \mathbf{r}_0|) + \tag{8}$$

$$+\frac{1}{4\pi} m_{jk}\partial_i\partial_j\partial_k \left[F_1(|\mathrm{r} - \mathrm{r}_0|) - F_2(|\mathrm{r} - \mathrm{r}_0|)\right].$$

For a fluid, where $c_2 = 0$ ($\mu = 0$) and $m_{ij} = -m\delta_{ij}$, this solution becomes

$$\mathbf{u}^p = -\frac{m}{4\pi c_1^2} grad \frac{\cos k_1 |\mathbf{r} - \mathbf{r}_0|}{|\mathbf{r} - \mathbf{r}_0|}. \tag{9}$$

To apply these results to a sphere we need to use expansions in a series of (orthogonal) vector spherical harmonics, defined by[10]

$$\mathbf{R}_{lm} = Y_{lm}\mathbf{e}_r\,,$$

$$\mathbf{S}_{lm} = \frac{\partial Y_{lm}}{\partial\theta}\mathbf{e}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi}\mathbf{e}_\varphi\,, \tag{10}$$

$$\mathbf{T}_{lm} = \frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi}\mathbf{e}_\theta - \frac{\partial Y_{lm}}{\partial\theta}\mathbf{e}_\varphi\,,$$

$l \neq 0$, where Y_{lm} are spherical harmonics and $e_{r,\theta,\varphi}$ are the spherical unit vectors. The functions $\mathbf{R}_{lm}$ $\mathbf{S}_{lm}$ are called spheroidal functions, while the functions T_{lm} are called toroidal functions. The series expansion reads

$$\mathbf{u}^p = \sum_{lm}(f_{lm}^p \mathbf{R}_{lm} + g_{lm}^p \mathbf{S}_{lm} + h_{lm}^p \mathbf{T}_{lm})\,, \tag{11}$$

where f_{lm}^p, g_{lm}^p and h_{lm}^p are functions only of the radius r. A similar series holds also for the free solution $\mathbf{u}^f$ of equation (1).

The explicit form of the coefficients f_{lm}^p, g_{lm}^p and h_{lm}^p is extremely cumbersome. We prefer to work formally with equation (1) and series expansions of the full solution $\mathbf{u} = \mathbf{u}^p + \mathbf{u}^f$ and the force $\mathbf{F}(\omega)$, with coefficients f_{lm}, g_{lm}, and h_{lm} and $F^{r,s,t}$, respectively. Making use of such series expansions and the properties of the vector spherical harmonics,[10] we get the equations

$$f'' + \frac{2}{r}f' + \frac{\rho\omega^2}{\lambda + 2\mu}f - \left[2 + \frac{\mu l(l+1)}{\lambda + 2\mu}\right]\frac{1}{r^2}f +$$

$$+\frac{(\lambda + 3\mu)l(l+1)}{(\lambda + 2\mu)r^2}g - \frac{(\lambda + \mu)l(l+1)}{(\lambda + 2\mu)r}g' = -\frac{F^r}{\lambda + 2\mu},$$

$$g'' + \frac{2}{r}g' + \frac{\rho\omega^2}{\mu}g - \frac{(\lambda + 2\mu)l(l+1)}{\mu r^2}g + \tag{12}$$

$$+\frac{2(\lambda + 2\mu)}{\mu r^2}f + \frac{\lambda + \mu}{\mu r}f' = -\frac{F^s}{\mu},$$

$$h'' + \frac{2}{r}h' + \frac{\rho\omega^2}{\mu}h - \frac{l(l+1)}{r^2}h = -\frac{F^t}{\mu}\,,$$

where, for the sake of simplicity, we dropped out the suffixes lm.

We turn now to the boundary conditions. The force P acting

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(inwards) on the surface $r = R$ of the sphere, where R is the radius of the sphere, with the spherical components P_α = (α = r, θ ,φ) is $P_\alpha = n_\beta\sigma_{\alpha\beta} = \sigma_{\alpha r}$, where the stress tensor is given by $\sigma_{\alpha\beta} = 2\mu u_{\alpha\beta} + \lambda u_{\gamma\gamma}\delta_{\alpha\beta}$; we get

$$2\mu u_{\theta r} = P_\theta\,, 2\mu u_{\varphi r} = P_\varphi\,,$$
$$2\mu u_{rr} + \lambda div\mathbf{u} = P_r\,, \tag{13}$$

where *divu* is written in spherical coordinates,

$$div\mathbf{u} = \sum_{lm}\left[\frac{1}{r^2}\frac{d}{dr}(r^2 f_{lm}) - \frac{g_{lm}}{r}l(l+1)\right]Y_{lm} \tag{14}$$

(by using the properties of the vector spherical harmonics equations[10]). We compute the strain tensor $u_{\alpha\beta}$ in spherical coordinates[7]

$$u_{rr} = \frac{\partial u_r}{\partial r}, u_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, u_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r},$$

$$2u_{\theta\varphi} = \frac{1}{r}\left(\frac{\partial u_\varphi}{\partial \theta} - u_\varphi\cot\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \varphi}, 2u_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta},$$

$$2u_{\varphi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} \tag{15}$$

by using the spherical components

$$u_r = \sum_{lm} f_{lm}Y_{lm}\,,$$

$$u_\theta = \sum_{lm} g_{lm}\frac{\partial Y_{lm}}{\partial \theta} + \sum_{lm}\frac{h_{lm}}{\sin\theta}\frac{\partial Y_{lm}}{\partial \varphi}\,, \tag{16}$$

$$u_\varphi = \sum_{lm}\frac{g_{lm}}{\sin\theta}\frac{\partial Y_{lm}}{\partial \varphi} - \sum_{lm} h_{lm}\frac{\partial Y_{lm}}{\partial \theta}$$

of the expansion of the displacement vector and the definition of the vector spherical functions (equations (10)). Similarly, we decompose the force *P* in vector spherical harmonics (with coefficients $P^{r,s,t}$) and identify its spherical components. The boundary conditions given by equations (13) lead to

$$2\mu f' + \lambda\left[\frac{2}{r}f + f' - \frac{g}{r}l(l+1)\right]|_{r=R} = P^r\,,$$

$$\mu\left(\frac{g}{r} - g' - \frac{f}{r}\right)|_{r=R} = -P^s\,, \tag{17}$$

$$\mu\left(\frac{h}{r} - h'\right)|_{r=R} = -P^t\,,$$

where we dropped the subscripts *lm*.

Vibration eigenfrequencies for large radius

The solutions *f*, *g* and *h* of equations (12) consist of free solutions (solutions of the homogeneous equations (12)) plus particular solutions. The homogeneous third equation (12), which describes toroidal vibrations, is the equation of the spherical Bessel functions $j_l(kr), k = \sqrt{\rho\omega^2/\mu} = \omega/c_2$. For F^t = 0 and P^t = 0 (a free surface) the third equation in the boundary conditions (17) gives

$$j_l(kR) = kRj_l'(kR)\,; \tag{18}$$

this equation has an infinity of solutions β_{ln}, labeled by an integer *n*, such that we get the eigenfrequencies

$$\omega_{ln} = \frac{c_2}{R}\beta_{ln}\,. \tag{19}$$

We can get an approximate estimate of the numbers β_{ln} by using the asymptotic expression of the spherical Bessel functions[11]

$$j_l(kr) \simeq \frac{1}{kr}\cos\left[kr - (l+1)\frac{\pi}{2}\right], kr \gg 1\,; \tag{20}$$

For $kR \gg 1$ equation (18) becomes

$$\tan\left[kR - (l+1)\frac{\pi}{2}\right] = -\frac{2}{kR}\,, \tag{21}$$

which have the approximate zeroes

$$\beta_{ln} \simeq n\pi + (l+1)\frac{\pi}{2}\,, \tag{22}$$

where *n* is any (large) integer. We can see that the frequencies are dense for large $R(\Delta\omega_{ln} = \pi c_2/R)$. The free toroidal solution is a superposition of $j_l(k_{ln}r)$, where $k_{ln} = \omega_{ln}/c_2 = \beta_{ln}/R$, with undetermined coefficients.

In general, for $F^t \neq o$ and $P^t \neq o$ the free toroidal solution is $C_l j_l(kr)$, where the constants C_l are determined from the boundary condition. It is easy to see that these coefficients include singular factors proportional to $\sim\frac{1}{\omega - \omega_{ln}}$, such that, the integration over frequencies leads to toroidal vibrations governed by the eigenfrequencies ω_{ln}. In general, the solution h_{lm} depends on two integration constants, which are determined by the boundary condition and the condition of a finite solution at the origin.

We pass now to the spheroidal components which involve the functions *f g* in equations (12) and (17). We note that the two coupled equations (12) are for the functions *f* and *g* include Bessel operators for spherical Bessel functions. We can get a simplified picture of these equations for large values '*r*'. Indeed, it is easy to see that in the limit $\omega r / c_{1,2} \gg l^2$ the free solutions are

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$$f \simeq \frac{A}{r}\cos(\omega r/c_1+\varphi_{1l}),\, g \simeq \frac{B}{r}\cos(\omega r/c_2+\varphi_{2l}), \quad (23)$$

where the coefficients A and B and the phases $\varphi_{1l,2l}$ remain undetermined. In this limit the boundary conditions are $f'|_{r=R} = g'|_{r=R} = 0$ and the eigenfrequencies are given by $\omega_{nl}R/c_{1,2}+\varphi_{1l,2l} = n\pi$, where n is any integer (the roots of the equation $\sin(\omega R/c_{1,2}+\varphi_{1l,2l}) = 0$).

The condition $\omega r / c_{1,2} \gg l^2$ is satisfied for a large r and a reasonably large range of frequencies and parameter l. For instance, for r those close to the Earth's surface, which is the spatial region of interest, we get $\omega R / c$ the order $\simeq 10^3\omega$ for the mean radius of the Earth R = 6370 km and the mean velocity of the elastic waves c = 5 km / s. Indeed, frequencies as low as $\omega = 10^{-3}\,S^{-1}$ are known for Earth's seismic vibrations [12-14].

We can see that there are two branches of spheroidal eigenfrequencies (corresponding to the velocities c_{12}), which are dense (continuous) for large R, very similar to the infinite space (as expected for large R); the $\omega^{(2)}$ -branch, although close to the toroidal branch, is distinct (there is a total of three branches of eigenfrequencies, corresponding to the three degrees of freedom; in the limit of the rotations of the sphere as a whole their frequencies go to zero (acoustic modes)). For non-vanishing forces, we have spheroidal vibrations driven by these forces, as discussed in the previous cases. The set of all eigenfrequencies is called the (seismic) spectrum. Earth's eigenmodes with eigenfrequencies of the order 10^{-3} - $10^{-4}\,S^{-1}$, excited by earthquakes, have been discussed in Refs [12-14].

The numerical solution of equations (12) indicates that the lowest mode (the fundamental mode) is S_{lm} with l = 2 and n = 0 (therefore, we may denote it as $S_{l=2,m}^{(n=0)}$);[6] it is denoted by ${}_0S_2$, and its eigenfrequency is denoted ω_{20}; the corresponding period is approximately 1 an hour. Much later, the Earth's crust was modeled as a series of superposed layers, with welded interfaces; the vibrations of such a stack of layers can be computed and long periods of the fundamental modes have been obtained; the dispersion relation of these modes (*i.e.*, the dependence of the frequency on their label n) can give information about the inner crustal structure [15,16] The first observation of " free oscillations of the Earth as a whole" was made for the Kamchatka earthquake of November 4, 1952;[17] they were followed by many observations of the Earth's vibrations caused by the great Chile earthquake of May 22, 1960 [18-20] (with magnitude greater than 8, which saturated the scales[21]). Today, eigenoscillations of the Earth can be recorded even for small earthquakes [22].

From studies of propagation of the seismic waves, it was inferred the Earth's solid inner core [23,24] of radius $\simeq$1000km and the outer liquid core of radius radius $\simeq$2000km. The inner-outer core discontinuity is called the Bullen, or Lehmann, discontinuity. The temperature of the inner core is radius $\simeq$6000km (iron and nickel) and the pressure is $\simeq 10^{12}$ dyn/cm^2. The buoyancy at this boundary could be the source of convection currents that generate the Earth's magnetic field (geodynamo effect). The next layers are a viscous mantle of thickness $\simeq$3000km and the solid crust of thickness$\simeq$70km. The boundary between mantle and crust is known as the Mohorovic discontinuity.

Fluid sphere

For a fluid sphere, the shear modulus μ is zero (μ= 0); equations (12) become

$$f'' + \frac{2}{r}f' + k^2 f - \frac{2}{r^2}f - \frac{d}{dr}\left[\frac{l(l+1)g}{r}\right] = -\frac{F^r}{\lambda}$$

$$\frac{1}{r}f' + \frac{2}{r^2}f - \frac{l(l+1)}{r^2}g + k^2 g = -\frac{F^s}{\lambda}, \quad (24)$$

where $k^2 = \rho\omega^2/\lambda = \omega^2/c^2$; the boundary condition reads

$$\left[\frac{2}{r}f + f' - \frac{g}{r}l(l+1)\right]|_{r=R} = \frac{P^r}{\lambda}. \quad (25)$$

Let us introduce $divu$, given by equation (14), which includes

$$d = f' + \frac{2}{r}f - \frac{g}{r}l(l+1). \quad (26)$$

Then the boundary condition becomes

$$d|_{r=R} = \frac{P^r}{\lambda}, \quad (27)$$

the second equation (24) reads

$$\frac{d}{r} + k^2 g = -\frac{F^s}{\lambda} \quad (28)$$

and the first equation (24) is

$$d' + k^2 f = -\frac{F^r}{\lambda}. \quad (29)$$

Hence, we have

$$g = -\frac{d}{k^2 r} - \frac{F^s}{\lambda k^2},\, f = -\frac{d'}{k^2} - \frac{F^r}{\lambda k^2}. \quad (30)$$

Now we introduce these functions in equation (26) and get

$$d'' + \frac{2d'}{r} + k^2 d - \frac{l(l+1)}{r^2}d = -\frac{(F^r)'}{\lambda} - \frac{2F^r}{\lambda r} + \frac{F^s}{\lambda r}l(l+1). \quad (31)$$

For free vibrations this is the Bessel equation for spherical Bessel functions $d = j_l(kr)$; the boundary condition (27) leads to the eigenfrequencies ω_{ln} = $(c/R)\beta_{ln}$, $j_l(\beta_{\lambda\nu})$ = 0. In a fluid, we have only pressure p, and the stress tensor is $\sigma_{ij} = p\delta_{ij}$ ($\sigma_{ij} = 2\mu u_{kk}\delta_{ij}$) with μ = 0); therefore, for a fluid $p = -\lambda u_{ii} = \lambda divu$; the equations written above for d are in fact equations for the pressure p. It is convenient to introduce the decomposition in Helmholtz potentials u $=grad\Phi + curl$/A, $divA$ = 0 and F = $grad\varphi + cur/h$, $divh$ = 0; then, $p = -\lambda\Delta\Phi$ and the equation of motion $\rho\ddot{\mathbf{u}} = \lambda grad\cdot div\mathbf{u} + \mathbf{F} = -gradp + \mathbf{F}$ becomes $\rho\ddot{\Phi} = \lambda\Delta\Phi + \varphi$

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, where the potential φ is given by $\Delta\varphi = divF$ and $h = 0, A = 0$. For vibrations this equation reads $c^2\Delta\Phi + \omega^2\Phi = -\frac{1}{\rho}\varphi$ and for F = $Mgrad\delta(r-r_o)$ we get $\varphi = -M\delta(r-r_0), \Phi = -m\frac{\cos k|r-r_0|}{4\pi c^2|r-r_0|}$ ($m = M/\rho$) and the solution u^p given by equation (9).

Static self-gravitation

A gravitational force

$$FdV = G\rho dV\frac{m}{r^2} = \frac{4\pi}{3}G\rho^2 rdV \tag{32}$$

acts upon a volume element dV placed at a distance r from the center of a sphere, where $G = 6.67\times 10^{-8}\, cm^3/\, g\cdot s^2$ is the universal constant of gravitation, ρ is the density of the sphere (assumed incompressible), and $m = (4\pi/3)\rho r^3$ is the mass of the sphere with radius r. If the sphere is compressible, the gravitational potential φ is given by the Poisson equation $\Delta\varphi = 4\pi G\rho$ and the gravitational force per unit mass is $F = -grad\varphi$; the condition of (hydrostatic) equilibrium (for a non-rotating sphere) reads $grad\rho = \rho F = -\rho grad\varphi$, such that $div[(grad\rho)/\rho] = -4\pi G\rho$; the dependence of the pressure on the density is given by the equation of state; for a constant density the pressure for a self-gravitating sphere of radius R at rest with free surface is $p = (2\pi/3)G\rho^2(R^2-r^2)$ (it seems that the pressure in the inner Earth's (solid) core is $\simeq 300GPa = 3\times 10^{12} dyn/cm^2$). Making use of equation (32), the equation of the elastic motion reads

$$\rho\ddot{\mathbf{u}} - \mu\Delta\mathbf{u} - (\lambda+\mu)grad\, div\mathbf{u} = \mathbf{F} = -\gamma\mathbf{r}\,, \tag{33}$$

where $\gamma = (4\pi/3)G\rho^2$. Since $Y_{00} = 1/\sqrt{4\pi}$ we may write

$$\mathbf{F} = -\gamma\mathbf{r} = -\sqrt{4\pi}\gamma r Y_{00}\mathbf{e}_r\,, \tag{34}$$

whence we can see that F an expansion is series of spheroidal and toroidal functions with all the coefficients zero, except the coefficient $F_{00}^r = -\sqrt{4\pi}\gamma r$ of the function R_{oo}; it follows that the motion may include all the eigenmodes S_{lm} and T_{lm}, as well as all the eigenmodes R_{lm}, the latter with $l \neq 0$; for $l = 0$, $m = 0$the motion, described by $f = f_{oo}$, is driven by the gravitational force. We note also that the force in equation (33) is static, which means that its Fourier transform is proportional to $\delta(\omega)$. For $l = 0$ the first equation (12) includes only the function f, *i.e.* $f\,\delta(\omega)$; this equation reads

$$f'' + \frac{2}{r}f' - \frac{2}{r^2}f = \frac{\sqrt{4\pi}\gamma}{\lambda+2\mu}r\,. \tag{35}$$

It is easy to see that a particular solution of this equation is $[\sqrt{4\pi}\gamma/10(2\mu+\lambda)]r^3$, while the homogeneous part of this equation has the solution $C_1 r + C_2/r_2$, where $C_{1,2}$ there are constants of integration; we must take $C_2 = 0$, it because the solution is finite at the origin. We are left with the solution

$$u_r = Ar^3 + C_1 r\,, A = \frac{\gamma}{10(2\mu+\lambda)}. \tag{36}$$

This solution must satisfy the boundary conditions at the surface of the sphere; making use of equations (16), we have the strain tensor $u_{rr} = u_r'$ and $u_{\theta\theta} = u_{\varphi\varphi} = u_r/r$; the force on the surface is $-\sigma_{\alpha r}|_R$, where the stress tensor is given by $\sigma_{\alpha\beta} = 2\mu u_{\alpha\beta} + \lambda u_{\gamma\gamma}\delta_{\alpha\beta}$; for a free surface we get the boundary condition

$$(2\mu+\lambda)u_r' + 2\lambda\frac{u_r}{r}|_{r=R} = 0 \tag{37}$$

($\sigma_{\alpha r}|_R = 0$), whence we determine the constant $C_1 = -[(6\mu+5\lambda)/(2\mu+3\lambda)]AR^2$ and, finally, the radial displacement

$$u_r = Ar\left(r^2 - \frac{6\mu+5\lambda}{2\mu+3\lambda}R^2\right) = \frac{\gamma}{10(2\mu+\lambda)}r\left(r^2 - \frac{6\mu+5\lambda}{2\mu+3\lambda}R^2\right); \tag{38}$$

we note that the radial displacement u_r is negative, as expected. It is worth estimating the radial displacement at the surface due to gravitation

$$u_r|_{r=R} = -\frac{\gamma}{5(2\mu+3\lambda)}R^3\,; \tag{39}$$

making use of $\rho = 5g/cm^3, \lambda, \mu \simeq 10^{11} dyn/cm^2$ (parameters for Earth), we get $\gamma \simeq 10^{-6}\, g/cm^3s^2$ and $u_r|_R \simeq 10^{-18}R^3 cm \simeq 10^8 cm = 10^3 km$, for the Earth's radius $R \simeq 6\times 10^8\, cm$; this is a distance of the order of the Earth's radius. Moreover, the strain is of the order 1/6, which may cast doubts on the validity of the linear elasticity used in this estimation. In addition, we note that the density suffers an important change due to the static gravitational field. Indeed, the change in density is $\delta\rho = -div(\rho u) = -\rho_o divu$, where ρ_o is the uniform initial density; with u given by equation (38) we get

$$\frac{\delta\rho}{\rho_0} = A(3\alpha R^2 - 5r^2)\,, \alpha = \frac{6\mu+5\lambda}{2\mu+3\lambda}\,, \tag{40}$$

which is of the order unity. The proper estimation of the static effect of the self-gravitational field on the elastic sphere is to solve simultaneously the equation of elastic equilibrium (33) with $F = -\rho grad\varphi$ and the Poisson equation for the gravitational field φ, $\Delta\varphi = 4\pi G\rho$. With spherical symmetry we have

$$\mathbf{F} = -\frac{4\pi}{3r^2}G\rho\int_{0<r'<r}d\mathbf{r}'\rho\frac{\mathbf{r}}{r}; \tag{41}$$

the Poisson equation for the gravitational potential may be written as $\Delta(F/\rho) = -4\pi Ggrad\rho$, such that the problem involves two equations and unknowns, u and ρ. Since this is a more difficult problem it is preferable to consider the density ρ as an empirically known function of r (a parametrization in powers of r can be used for ρ and a variational approach can be

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applied to the problem). Even so, the equations governing the influence of the gravitational field upon the elasticity of a self-gravitating sphere are difficult.

Dynamic self-gravitation

Let us assume a spheric, non-rotating, homogeneous, elastic Earth at equilibrium under the action of its gravitational field; we consider small elastic deformations of this equilibrium state; in the first approximation, we have a small change denoted by K in the gravitational potential as a consequence of the small changes in density $div(\rho u)$, *i.e.*, we have

$$\Delta K = -4\pi G div(\rho \mathbf{u}) , \quad (42)$$

where ρ is a known function of r. The equation of elastic motion reads

$$\rho\ddot{\mathbf{u}} - \mu\Delta\mathbf{u} - (\lambda+\mu) grad\, div\mathbf{u} = -\rho gradK . \quad (43)$$

These two coupled (vectorial) equations are difficult to be treated by an analytical method, due to the non-uniformity of the density. For a uniform density, taking the *div* in equation (43) and using equation (42) we get for D = *divu*

$$\rho\ddot{D} - (\lambda+2\mu)\Delta D = 4\pi G\rho^2 D , \quad (44)$$

an equation which indicates that the frequency ω changes by

$$\Delta(\omega^2) = -4\pi G\rho ; \quad (45)$$

for frequencies as low as $\omega = 10^{-4}\ S^{-4}$ the variation given by equation (45) is large. Let us use the Helmholtz decomposition $u = grad\Phi + curlA$, $divA = 0$; then, from equation (42) we have $K = -4\pi G\rho\Phi$ and from equation (43) we get $\Delta\Phi + k_1^2\Phi = 0, \Delta\mathbf{A} + k_2^2\mathbf{A} = 0$. These are the same equations as those which hold in the absence of the gravitational field, except that k_1^2 is changed into $k_1^2 \to k_1^2 + 4\pi\rho G / c_1^2$. Moreover, we can see that only the spheroidal modes are affected by gravitation (since $div\mathrm{T}_{lm} = 0$). It follows that the spheroidal frequencies (*i.e.*, the branches $\omega^{(1,2)}$) are given by the same relations of the type $\omega = (c/R)\beta$, where denote the zeros the spherical Bessel functions in the limit of large$R$; for c $=c_1$, this relation reads $\omega^2 + 4\pi\rho G = (c_1^2 / R^2)\beta^2$. Hence, we may see that we should have the inequality $(c_1^2 / R^2)\beta^2 > 4\pi\rho G$, or $(\lambda+2\mu)\beta^2 > 4\pi\rho^2 GR^2$. The term on the right side of this inequality is, up to an immaterial numerical factor, the pressure due to the gravitation at the origin; it is much larger than the elastic pressure $\lambda + 2\mu$. The inequality is not satisfied for small values of β (as required by experimental observations). It follows that the model of an elastic solid Earth is not valid for the interior of the Earth. In those central regions, the elasticity is not able to sustain the gravitational pressure. Likely, an additional pressure exists there, which compensates for the gravitational pressure. The large dimensions of the mantle and liquid outer core complicate the matter, and such an Earth's model may exhibit very low frequencies (undertones);[25] If so, we may leave aside the effects of the gravitation in estimating the elastic vibrations of the Earth. In this case, c= 5km/s we get a period $T \simeq (2.2/\beta)$ of hours; the smallest zero of j_2 (corresponding approximately to the mode ${}_0S_2$) is ≃= 3.6;[11] we get $T \simeq 37$ minutes (for a velocity c = 3km/s the period is T = 61 minutes, which agrees with the experimental observations).

Rotation effect

If a vector a rotates, its change is $\delta a + \alpha \times a$, where $\delta\alpha$ is the infinitesimal rotation angle; therefore, its velocity is $a + \Omega\times a$, where Ω is the angular velocity; its acceleration is $\ddot{\mathbf{a}} + \dot{\mathbf{\Omega}}\times\mathbf{a} + 2\mathbf{\Omega}\times\dot{\mathbf{a}} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathrm{a})$. Let us apply this relation to the displaced position a = r + u; we get the acceleration $\ddot{\mathbf{u}} + \dot{\mathbf{\Omega}}\times(\mathbf{r}+\mathbf{u}) + 2\mathbf{\Omega}\times\dot{\mathbf{u}} + \mathbf{\Omega}\times[\mathbf{\Omega}\times(\mathrm{r}+\mathrm{u})]$; we can see that additional forces appear in rotation: $-2\mathbf{\Omega}\times\dot{\mathbf{u}}$ is the Coriolis acceleration and $-\Omega\times[(r + u)]$ is the centrifugal acceleration. The Earth rotates with a constant angular velocity $\Omega = 2\pi/T$, T = 24 hours, oriented along the z -axis. We write the equation of elastic motion as

$$\rho\ddot{\mathbf{u}} + 2\rho\mathbf{\Omega}\times\dot{\mathbf{u}} = \mathbf{F} , \quad (46)$$

where F includes the elastic force (*i.e.*, $Fi = \partial_j\sigma_{ij}$) and other external forces, and the centrifugal force is omitted since Ω is much smaller than the eigenfrequencies of the Earth (an estimation of the longest periods of the Earth's eigenmodes gives an order of magnitude $2\pi R/c\beta \simeq 37$ minutes, for the wave velocity c = 5km/s and β = 3.6 where R is the Earth's radius).

In the absence of the Coriolis force in equation (46) we decompose the force F and the displacement u in normal modes by using the spheroidal and toroidal functions. Let us focus on one normal mode, for instance, a toroidal mode $\mathbf{u}_{lm}^{(n)} = h_l^{(n)}\mathbf{T}_{lm}$, corresponding to the eigenfrequency ω_{ln} = $(c_2/R)\beta_{ln}$, where β_{ln} is, approximately, a zero of the function $j_l(k^{(n)}R)$; the eigenfunctions $h_l^{(n)}$ are given by the spherical Bessel functions $j_l(k^{(n)}r)$; it is preferable to multiply these functions by constants and fix these constants such as

$$\int dr \cdot r^2 h_l^{(n)}(r) h_l^{(n')}(r) = \delta_{nn'} ; \quad (47)$$

we recall that the toroidal functions are orthogonal, *i.e.*

$$\int do \mathbf{T}_{lm}\mathbf{T}_{l'm'}^{*} = \delta_{ll'}\delta_{mm'} . \quad (48)$$

Since $\Omega/\omega_{ln} \ll 1$ we solve equation (46) by a perturbation-theory method. First, we drop the labels l,m and n and use the notations $\mathbf{u}_{lm}^{(n)} = \mathbf{u}_0$, $\omega_{lm} = \omega_o$; we seek the solution as a series in powers of Ω/ω_o

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$$\mathbf{u}=\mathbf{u}_0+\frac{\Omega}{\omega_0}\mathbf{u}_1+\dots, \tag{49}$$

where u_1 to be determined, is assumed orthogonal on u_0,[26] with respect to the scalar product defined as the integration over the whole space, *i.e.*

$$\int d\mathbf{r}\mathbf{u}_1\mathbf{u}_0=0. \tag{50}$$

A similar series is valid for the frequency

$$\omega=\omega_0+\frac{\Omega}{\omega_0}\omega_1+\dots. \tag{51}$$

Introducing these series in equation (46), with time Fourier transforms, we get

$$-\rho\omega_0^2\mathbf{u}_0=\mathbf{F},$$
$$-\rho\omega_0\Omega\mathbf{u}_1-2\rho\Omega\omega_1\mathbf{u}_0-2i\rho\omega_0\mathbf{\Omega}\times\mathbf{u}_0=0; \tag{52}$$

the first equation (52) defines the function u_0; in the second equation (52) we take the scalar product with u_0 and use the orthogonality of u_0 with u_1 we get

$$\omega_1=-\frac{i\omega_0}{l(l+1)}\int d\mathbf{r}e_z(\mathbf{u}_0\times\mathbf{u}_0^*), \tag{53}$$

where we put $\Omega=\Omega e_z$, e_z being the unit vector along the z -axis. Here we use $e_z=\cos\theta e_r-\sin\theta e_\theta$, $\mathbf{u}_0=h_l^{(n)}\mathbf{T}_{lm}$ and T_{lm} from equations (10); we get immediately

$$\omega_1=\omega_0\frac{m}{l(l+1)}, \tag{54}$$

where m denotes all integers from $-l$ to l. It follows that the frequencies ω_{ln}, which are degenerate with respect to m, are split into $2l+l$ branches

$$\omega_{ln}\to\omega_{ln}+\Omega\frac{m}{l(l+1)}; \tag{55}$$

using ω_1 thus determined, we can get u_1 from the second equation (52). Higher-order contributions can be obtained similarly. An m -band occurs for each ω_{ln}, of the widths $2\Omega/(l+1)$, with the separation frequency $\Omega/(l+1)$. For a typical eigenperiod 60 minutes the ratio Ω/ω_0 is approximately $1/20\ll1$.

Centrifugal force

The equation of the elastic motion for a body in rotation with a (constant) angular velocity Ω reads

$$\rho\ddot{\mathbf{u}}+2\rho\mathbf{\Omega}\times\dot{\mathbf{u}}+\rho\mathbf{\Omega}\times[\mathbf{\Omega}\times(\mathbf{r}+\mathbf{u})]=\mu\Delta\mathbf{u}+(\lambda+\mu)grad\,div\mathbf{u}+\mathbf{F}, \tag{56}$$

where F is an external force. We note that the centrifugal term ⊠$\Omega\times(\Omega\times r)$ is static, so we can write it as

$$\mathbf{F}_c=\rho\mathbf{\Omega}(\mathbf{\Omega}\mathbf{r})-\rho\Omega^2\mathbf{r}, \tag{57}$$

where we denoted by F_c the centrifugal force and removed any other external force (F=0); we may neglect u in the centrifugal force, since it is very small in comparison to r. The angular velocity is oriented along the z -axis, $\Omega=\Omega e_z$. Making use of $e_z=\cos e_r-\sin\theta e_\theta$ and the spherical harmonics

$$Y_{00}=\frac{1}{\sqrt{4\pi}},Y_{20}=\sqrt{\frac{5}{16\pi}}(1-3\cos^2\theta), \tag{58}$$

it is easy to see that we can write F_c as a series expansion

$$\mathbf{F}_c=-\rho\Omega^2r\left(\alpha\mathbf{R}_{00}+2\beta\mathbf{R}_{20}-\beta\mathbf{S}_{20}\right) \tag{59}$$

in spheroidal functions, where $\alpha=2\sqrt{4\pi}/3$ and $\beta=\sqrt{16\pi/5}$. We seek a similar expansion for the displacement u,

$$\mathbf{u}=f_1\mathbf{R}_{00}+f_2\mathbf{R}_{20}+g\mathbf{S}_{20}; \tag{60}$$

equations (12) lead to

$$f_1''+\frac{2}{r}f'-\frac{2}{r^2}f_1=-\frac{\rho\Omega^2\alpha}{\lambda+2\mu}r,$$

$$f_2''+\frac{2}{r}f_2'-\frac{2(\lambda+5\mu)}{\lambda+2\mu}\frac{1}{r^2}f_2+\frac{6(\lambda+3\mu)}{\lambda+2\mu}\frac{1}{r^2}g-\frac{6(\lambda+\mu)}{\lambda+2\mu}\frac{1}{r}g'=-\frac{2\rho\Omega^2\beta}{\lambda+2\mu}r,$$

$$g''+\frac{2}{r}g'-\frac{6(\lambda+2\mu)}{\mu}\frac{1}{r^2}g+\frac{2(\lambda+2\mu)}{\mu}\frac{1}{r^2}f_2+\frac{\lambda+\mu}{\mu}\frac{1}{r}f_2'=-\frac{\rho\Omega^2\beta}{\mu}r. \tag{61}$$

We seek solutions of these equations of the form $f_{1,2},g=Ar^n$; the solution of the homogeneous equations (regular in the origin) corresponds to $n=1$; we get

$$f_1=-\frac{\rho\Omega^2\alpha}{10(\lambda+2\mu)}r^3+C_1r \tag{62}$$

and

$$f_2=C_2r,\ g=\frac{\rho\Omega^2\beta}{6\lambda}r^3+C_3r, \tag{63}$$

where $C_{1,2,3}$ are constants of integration. These constants are determined by the boundary conditions given by equations (17) for a free surface. Finally, we get the displacement

$$\mathbf{u}=-\frac{\rho\Omega^2}{3\lambda}\left[\frac{\lambda}{5(\lambda+2\mu)}r\left(r^2-\frac{5\lambda+2\mu}{3\lambda+2\mu}R^2\right)-R^2r(1-3\cos^2\theta)\right]\mathbf{e}_r+$$
$$+\frac{\rho\Omega^2}{3\lambda}r\left[r^2-\frac{2(3\lambda+\mu)}{3\lambda}R^2\right]\sin\theta\cos\theta\mathbf{e}_\theta. \tag{64}$$

It is worth estimating the equatorial displacement ($\theta=\pi/2$) for the Earth radius $R=6370km$; with $\rho=5g/cm^3$ and $\lambda,\mu=10^{11}dyn/cm^2$ we get $u=u_r\simeq10km$.

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Earthquake " temperature"

Let us multiply by u the equation of the elastic motion,

$$\rho\ddot{\mathbf{u}}+\mu curl\,curl\mathbf{u}-(\lambda+2\mu)grad\,div\mathbf{u}=\mathbf{F}\,; \tag{65}$$

integrating by parts, we get the law of energy conservation

$$\frac{\partial\mathcal{E}}{\partial t}=-div\mathbf{S}+w\,, \tag{66}$$

where

$$\mathcal{E}=\frac{1}{2}\rho\dot{\mathbf{u}}^2+\frac{1}{2}\mu(curl\mathbf{u})^2+\frac{1}{2}(\lambda+2\mu)(div\mathbf{u})^2 \tag{67}$$

is the energy density,

$$S_i=\mu(\dot{u}_j\partial_j u_i-\dot{u}_j\partial_i u_j)-(\lambda+2\mu)\dot{u}_i\partial_j u_j \tag{68}$$

are the components of the energy flux density and $w = uF$ is the density of mechanical work done by the external force per unit time. It is worth noting that the energy density given by equation (67) differs from the energy density derived from the other form of the equation of motion, *e.g.*,

$$\rho\ddot{\mathbf{u}}-\mu\Delta\mathbf{u}-(\lambda+\mu)grad\,div\mathbf{u}=\mathbf{F}\,, \tag{69}$$

by the divergence of a vector; it follows that the energy density and the energy flux density are not unique (well defined).

Making use of equations (6), (7) and (10) we can write symbolically

$$curl\mathbf{u}=\frac{h}{r}l(l+1)\mathbf{R}+\frac{1}{r}\frac{d}{dr}(rh)\mathbf{S}+\left[\frac{f}{r}-\frac{1}{r}\frac{d}{dr}(rg)\right]\mathbf{T}\,,$$

$$div\mathbf{u}=\frac{1}{r^2}\frac{d}{dr}(r^2 f)-\frac{g}{r}l(l+1)\,. \tag{70}$$

We compute the total energy E by introducing these expressions for $curl u$ and $div u$ in equation (67), integrating over the solid angle and integrating by parts over the radius r; for large values of R the boundary conditions given by equations (17) for free vibrations ensure the vanishing of the " surface terms" in the r-integration by parts; in addition, for large values of R we may neglect the f -term in $curl u$ and the g -term in $divu$; making use of the equations of motion (12), we get finally

$$E\simeq\frac{2l+1}{8\pi}\int d\mathbf{r}\rho\omega^2\left[f^2+l(l+1)g^2+l(l+1)h^2\right], \tag{71}$$

where the summation over l is omitted (the factor $2l + 1$ arises from the summation over m). The functions ωf, ωg and ωh in equation (71) are superpositions of their own normal modes (labeled by n); for large values of R all these eigenmodes may be taken as the spherical Bessel functions, and the eigenfrequencies are given by the zeros of the derivatives of the spherical Bessel functions; we note that these eigenmodes are orthogonal concerning the r -integration; the f -part in equation (71) is related to the velocity c_1 (the combination of $\lambda + 2_\mu$ of the elastic moduli), while the g - and h -parts are related to the velocity c_2 (modulus μ).

Let us write the energy given by equation (71) for the normal modes as

$$E\simeq\frac{1}{8\pi}\sum_{lmn}\int d\mathbf{r}\left[\rho\omega_{ln}^{(r)2}f_{ln}^2+\rho\omega_{ln}^{(s)2}g_{ln}^2+\rho\omega_{ln}^{(t)2}h_{ln}^2\right], \tag{72}$$

where the summation over m is restored and the coefficients $l(l+1)$ are included in g_{ln} and h_{ln}. We may use approximately the asymptotic expressions for the functions f_{ln}, g_{ln} h_{ln} of the form f_{ln} = $a_{ln}\cos[kr-(l+1)\pi/2]/kr$ (spherical Bessel functions), with amplitudes a_{ln}; and, similarly, for g_{ln} and h_{ln} with amplitudes b_{ln} and, respectively, c_{ln} Effecting the integral, we get

$$E\simeq\frac{1}{4}R\sum_{lmn}\left[\rho c_1^2a_{ln}^2+\rho c_2^2b_{ln}^2+\rho c_2^2c_{ln}^2\right], \tag{73}$$

where R is the radius of the sphere and $c_{1,2}$ are the wave velocities. This is a simple expression, of the form

$$E=\sum_s\rho Rc^2a_s^2\,, \tag{74}$$

where s is a generic notation for the normal modes.

Let us assume that energy E is given to the vibrating sphere; we ask how it is distributed among the normal modes. It is reasonable to assume that, after many reflections from the surface, the distribution of energy reaches an equilibrium state, in the sense that it does not depend anymore on time. This state is characterized by a probability density w, which is multiplicative for different spheres; *In* w is additive, and the function

$$S=-w\ln w \tag{75}$$

should have a maximum value in the equilibrium state, corresponding to a maximal " disorder"; this represents our idea of equilibrium. Obviously, the function S given by equation (75) is the entropy. Its maximum value for constant energy is reached for the extremum of the function S $-\beta wE$, where β is a Lagrange multiplier; we get the Boltzmann (canonical) distribution

$$w=const\cdot e^{-\beta E}\,, \tag{76}$$

or, for one mode,

$$w=\sqrt{\beta\rho Rc^2/\pi}e^{-\beta\rho Rc^2a^2}\,. \tag{77}$$

The mean energy per mode is

$$\bar{e}=\frac{1}{2}T\,, \tag{78}$$

as expected, and the mean value of the square amplitude is

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$$\overline{a^2} = \frac{T}{2\rho Rc^2}, \quad (79)$$

where we introduced the temperature $T = 1/\beta$. The total mean energy is $\overline{E} = N\overline{e} = NT/2$, where N is the total number of modes; this equality gives the temperature parameter.

Making use of the asymptotic expressions of the spherical Bessel functions (for the radial functions) we get the normal modes given by $k_{ln}R = (2n+l+1)\pi/2$; hence, we see that the normal modes are equidistant; the corresponding wavelengths are $\lambda_{ln} = 4R/(2n+l+1)$. We may take, tentatively, a cutoff of short wavelengths of the order $10^{-4}\,cm$ ($1\mu m$, corresponding to a frequency $\simeq 5GHz$, for velocity $5km/s$); it is reasonable to admit that below this distance the homogeneous elastic qualities of the Earth do not hold anymore. For this cutoff, we get a maximum number $2n+l+1$ of the order $N_c = 10^{13}$ and several modes of the order $N = N_c^3 = 10^{39}$. Also, it is reasonable to assume that the earthquake energy is spent mainly on mechanical work (like fracture of the rocks, structural damage, etc). The largest part of the energy released in an earthquake is spent in mechanical work associated with the motion of the rocks, soil, and the damage produced at the Earth's surface; the remaining is dissipated as heat, after a long while. We may assume tentatively that only the energy $\overline{E} \simeq 10^{15.65} erg$ is spent in thermalization (corresponding to magnitude $M_w = 0$) in the Gutenberg-Richter law $\lg E = \frac{3}{2}M_w + 15.65$ (in erg). This way, we get a temperature $T = 10^{-23}\,erg$ (*i.e.*, $\simeq 10^{-7}K$, since $1.38 \times 10^{16}\,K = 1erg = 1dyn.cm$); the inner Earth's temperature is $\simeq 6000K$. The quantity ρRc^2 in equation (78) is $\rho Rc^2 \simeq 10^{20}\,g/s^2$ (for $\rho = 5g/cm^3$, $R \simeq 6\times10^8\,cm$ and $c = 5km/s$). The estimation of the temperature is very sensitive to the number of eigenmodes N, which may be much lower; also, we may allow for the thermalization of higher energies, corresponding to higher magnitudes. In both cases the temperature increases. We note that the cutoff wavelength, which affects essentially the numerical estimation of the temperature, corresponds to the mean inter-atomic distance in the Debye estimation of the statistical equilibrium of the elastic vibrations (phonons) in crystals.

Concluding remarks

A systematic analysis of vibrations of an elastic sphere is presented in this paper, to get analytical results. Such results are useful, on one hand, in analyzing the recorded data, and, on the other, in comparing them with the current numerical investigations. It is shown that the hypothesis of a large radius, appropriate for Earth's vibrations, simplifies the analysis to a great extent. We discuss the general formulation of the problem, the use of spherical harmonics, the approximation of a large radius, and the example of a fluid sphere. Specific results are given for toroidal and spheroidal vibrations. Also, the self-gravitation and rotation effects are analyzed in detail.

Apart from self-gravitation and rotation, the inhomogeneities may have an important effect on the vibrations of the solid sphere. For instance, from equation (1), a (uniform) change $\delta\rho$ in density causes a change $\delta\omega/\omega = -\delta\rho/2\rho$ in frequency. The effect of similar changes in the elastic moduli λ and μ can be estimated by using the changes in the wave velocities c in the relation $\omega_{ln} \simeq (c/R)\beta_{ln}$.

An approximate procedure is given in this paper for estimating the spectrum of eigenfrequencies (and eigenfunctions) of the vibrations of a solid sphere, with application to Earth's vibrations, as those produced by an earthquake. The procedure is sufficiently convenient to apply to other, more complex situations involving the vibrations of a solid sphere, as, for instance, the corrections brought about by self-gravitation, rotation, and inhomogeneities. The distribution of the energy among the vibrations eigenmodes is also estimated here and the concept of the earthquake " temperature" is tentatively introduced, as another means of characterizing earthquakes and estimating the earthquake's effects.

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